Translated Preface of “On the Addition of Fractions” by Petri Mengoli

Ivan D. Montiel

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Abstract

This document provides an English translation of the preface of “On the Addition of Fractions” by Petri Mengoli. I also describe the context of the work, including Archimedes’ Quadrature of the Parabola, Mengoli’s argument on the addition of fractions, and the Basel problem.

1 Introduction

This document contains an English translation of the preface of “On the Addition of Fractions” by Petri Mengoli. In this preface, Mengoli reflects on Archimedes’ Quadrature of the Parabola and presents his argument on the addition of fractions. This leads to the conclusion that the Harmonic series diverges, which is a well-known result in mathematics today. The preface of this publication also contains a statement of what came to be known as the Basel problem, which was solved by Leonhard Euler in the 18th century.

This document starts with some context on Archimedes’ Quadrature of the Parabola, explores Petri Mengoli’s argument, and calls out the first instance of what came to be known as the Basel problem.

I present the translation of Mengoli’s work as it was not easy or readily available to find an English translation of this treatise that included the original statement of the Basel Problem. Jordan Bell and Viktor Blåsjö have translated the core argument of Mengoli’s work in their paper “Pietro Mengoli’s 1650 Proof that the Harmonic Series Diverges” [2], but the statement of the Basel problem is not included in their translation. I hope this document can serve as a reference for those interested in the history of mathematics and the work of Petri Mengoli.

2 Archimedes’ Quadrature of the Parabola

The start of Petri Mengoli’s preface to his work “On the Addition of Fractions” is a reflection on Archimedes’ Quadrature of the Parabola [1]. In this treatise, Archimede finds the area of a parabolic segment by dissecting the area into infinitely many triangles to form a geometric progression.

If we take a parabola and have a chord pass through it, we can form a triangle within the parabola with a given height h and width w. The remaining area of the parabolic segment can also be dissected into two triangles, where the height of the triangle is $h/4$ and the width is $w/2$.

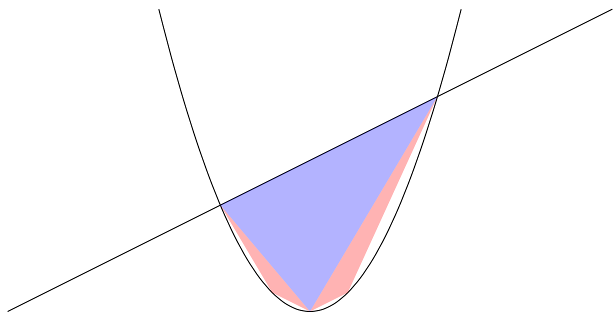

Figure 1: Archimedes' Quadrature of the Parabola

If the area of the initial triangle is T, then the area of the each if the smaller two triangles is $T/8$, and since there are two of them, the sum of the areas of the two smaller triangles is $T/4$.

We can continue this process using *the method of exhaustion* to see that as this sequence of triangles continues, we fill the area A:

$$A = T + \frac{T}{4} + \frac{T}{16} + \frac{T}{64} + \dots \tag{1}$$

Rearranging, the terms, we get:

$$A = T\left(1 + \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \dots\right) \tag{2}$$

And now we need only find what the sum of the geometric series converges to. Archimedes takes a geometric approach to find the sum of the series:

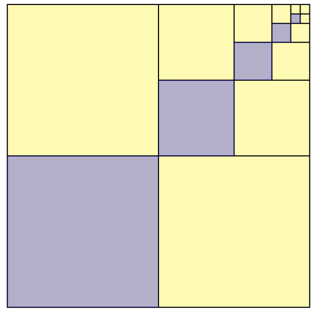

Figure 2: Archimedes' Geometric Approach to the Sequence

Each blue area is one-fourth of the area of the previous square. And since the squares are congruent to the two yellow squares next to it, the area the blue squares covers must be one-third of the total area. Hence,

$$\frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \ldots = \frac{1}{3} \tag{3}$$

From this, Archimedes concludes that the area of the parabolic segment is 4/3 times the area of the triangle, since we have an additional 1 term in the original sum. Hence,

$$A = \frac{4}{3}T \tag{4}$$

This is a remarkable result as it helps find the area of a parabolic segment without the use of calculus. It also does not rely on any concept of limits, or infinite series, but rather on the *method of exhaustion* to find the area of the parabolic segment.

3 Petri Mengoli's Argument

Petri Mengoli's work in "Novae quadraturae arithmeticae, seu De additione fractionum" (New Arithmetic Quadratures, or on the Addition of Fractions) is a treatise on the addition of fractions, in particular the harmonic series which in modern notation is:

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots \tag{5}$$

Mengoli's argument that the harmonic series diverges is the first known published proof of this result[2].

The argument can be summarized as follows: take the harmonic series and group the fractional terms, the first of which we look at is $1/2 + 1/3 + 1/4$. Mengoli shows that these three terms, when grouped, are greater than 1, which we can express more generally for any three terms as,

$$\frac{1}{n-1} + \frac{1}{n} + \frac{1}{n+1} > \frac{3}{n} \tag{6}$$

He then notes that this is also true when grouping the next 9 terms, the next 27 terms, the next 81 terms, and so on. Since we can always find another group of terms that sum to a value greater than 1, we can always find sets of groups that will exceed some value S. For example, we we assume that the harmonic series converges to 10, we can always find 11 groupings of terms using Mengoli's method such that the sum of the groupings exceeds 10. This leads to the conclusion that the sum of the harmonic series can exceed any value given a certain number of terms. Therefore, the harmonic series diverges.

The interesting part of Mengoli's argument is that he uses a similar approach of *the method of exhaustion* as Archimedes did in his Quadrature of the Parabola. This argument does not rely on the concept of limits, but rather on the properties of the fractions themselves. For this reason, Mengoli titled his work "New Arithmetic Quadratures" as it takes a similar approach to the methodologies of Archimedes, even though Mengoli does not introduce any new quadratures in his work.

4 The Basel Problem

From his conclusions on the harmonic series, Mengoli acknowledges that finding the value of the sum of the reciprocals of the squares "demands the assistance of greater ingenuity so that the precise sum"[4] of the series may be found.

This is the first known statement of what came to be known as the Basel problem, which is to find the value of the sum of the reciprocals of the squares:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots \tag{7}$$

The Basel problem was solved by Leonhard Euler in 1734 [3], who showed that the sum of the reciprocals of the squares is $\pi^2/6$, a result that came about from Euler comparing two infinite expansions of the sine function: the Taylor series expansion and the product expansion. The exact details of Euler's proof are beyond the scope of this document, but it is interesting to note that Mengoli's work led to the statement of the Basel problem, which was solved by Euler nearly a century later.

5 Preface of "On the Addition of Fractions" by Petri Mengoli

The following translation is based on the original text of "Novae quadraturae arithmeticae, seu De additione fractionum" by Petri Mengoli available via the Internet Archive[4].

Meditanti mihi persæpe Archimedis parabolae Quadraturam, propter quam infinita triangula in continuâ quadrupla proportione existentia certos limites quantitatis non excedunt; occurrit universalis illa Quadratura eiusdem argumenti occasione a Geometris demonstrata, qua magnitudines infinitæ continuam quamlibet proportionem maioris inaequalitatis possidentes in finitias homogeneas quantitates colliguntur. Admirabile sane Theorema: cuius contemplatione in eam quaestionem inductus sum, virum magnitudines ea quacunq lege dispositæ, ut aliqua possit assumi minor quolibet proposita, vel ut deficientes in infinitum evanescant, infinita composita omnem propositam quantitatem valeant superare.

While meditating on Archimedes' problem of the Quadrature of the Parabola, because of which infinite triangles arranged in continuous quadruple proportion exist without exceeding certain limits of quantity, that universal Quadrature of the same argument occurs on occasion as pointed out by the geometers, where infinite magnitudes possessing any continuous proportion of greater inequality are collected into finite homogeneous quantities. Truly an admirable theorem: by contemplating it, I was led into that question concerning magnitudes, arranged according to any given law, such that some may be assumed smaller than any proposed value, or such that those diminishing into infinity vanish, while infinite compositions surpass every proposed quantity.

In huiusmodi causae experimentum Arithmeticas fractiones tentare agressus, eas ita disposui, ut singulas unitates singulis post unitatem numeris denominaram, in qua quidem dispositione sumi potest magnitudo minor qualibet assignata, & propterea ipsæ magnitudines ad ordinis incrementum quantitate decrescentes in infinitum evanescunt.

In pursuit of an experiment concerning this type of problem, I attempted arithmetic fractions, arranging them such that each individual unit corresponds to each number following unity fractions, in which arrangement it is possible to take a magnitude smaller than any assigned value, and therefore the magnitudes themselves decrease in quantity with the progression of the order, vanishing into infinity.

$$\frac{1}{2}\;\frac{1}{3}\frac{1}{4}\;\frac{1}{5}\;\frac{1}{6}\;\frac{1}{7}\;\frac{1}{8}\;\frac{1}{9}\;\frac{1}{10}\;\frac{1}{11}\;\frac{1}{12}\;\frac{1}{13}\;\frac{1}{14} \tag{8}$$

Causam igitur in assumptæ dispositionis terminis proponens quærebam, utrum unitates denominatæ singulis numeris post unitatem in infinitum dispositæ, & aggregatæ infinitam aliquam, vel finitam compônerent extensionem. Pro finita extensione respondendum videbatur; quod numerorum, & fractionum contrariæ sint potestates, numerorum quidem in multiplicatione, qua magnitudines versus infinitum progrediuntur, fractionum vero in divisione, qua res ad ipsa indivisibilia reducitur: aggregati autem numeri superant quamlibet propositam quantitatem; ergo à contrario sensu aggregatæ fractiones non videntur posse quamlibet propositam magnitudinem excedere. Hoc sophisma toto ferè mense fuit exspectationis nis argumentum, quod pro hac parte Geometricam in causa ferre possem sen-

Therefore, proposing the cause within the terms of the assumed arrangement, I was questioning whether the units assigned to each number after unity, arranged into infinity, and aggregated, would form any infinite or finite extension. It seemed necessary to respond in favor of a finite extension; this is because the powers of numbers and fractions are opposite: numbers, indeed, in multiplication, by which magnitudes progress toward infinity, but fractions in division, by which the matter is reduced to indivisibles. Yet, the aggregated numbers surpass any given quantity; therefore, by contrary reasoning, it seems that aggregated fractions cannot exceed any proposed magnitude. This sophism occupied nearly an entire month's expectation of the argument, which on this matter I could render in favor of the getentiam: atqui dum processum demonstrationis examino, judicium in alterius partis favorem convertitur.

ometric cause. However, upon examining the progression of the demonstration, my judgment shifted in favor of the opposing side.

Ea est ratio, quia in propositis fractionibus æquales magnitudines numeris Arithmetice dispositis denominantur, & propterea tres consequentes, utpote A, B, C, sunt Harmonice dispositæ, &

The reason is as follows: in the proposed fractions, equal magnitudes are denominated by numbers arranged arithmetically. Therefore, the three consecutive terms, namely A, B, and C, are harmonically arranged, and

$$\begin{array}{ccc} A & B & C \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \end{array} \tag{9}$$

A, ad C, eandem habet proportionem, quam excessus A, B, ad excessum B, C: est autem A, maior C; ergo excessus A, B, maior est excessu B, C; & aggregatum A, C, maius duplo B; & aggregatum extremis A, B, C, maius triplo media B.

the proportion of A to C is the same as the proportion of the excess of A over B to the excess of B over C. Furthermore, A is greater than C; thus, the excess of A over B is greater than the excess of B over C. Moreover, the sum of A and C is greater than twice B, and the sum of the extremes A, B, and C is greater than three times the mean B.

Hoc igitur argumento fractiones in proposita dispositione sumpta terna a prima sunt maiores triplis mediis. Ergo fractiones propositæ dispositionis assumptæ totidem semper secundum numeros proportionis continuæ subtriplicæ 3, 9, 27, 81, singulas unitates excedunt.

Therefore, by this argument the fractions taken in the proposed arrangement in groups of three starting from the first, are larger than three times the mean. Therefore, the fractions of the proposed arrangement, when taken in the same quantity as the numbers of a continuous subtriplicate proportion 3, 9, 27, 81, each exceed unity.

$$\frac{1}{2}\ \frac{1}{3}\ \frac{1}{4} \quad \frac{1}{5}\ \frac{1}{6}\ \frac{1}{7} \quad \frac{1}{8}\ \frac{1}{9}\ \frac{1}{10} \quad \frac{1}{11}\ \frac{1}{12}\ \frac{1}{13} \quad \frac{1}{14}\ \frac{1}{15}\ \frac{1}{16} \tag{10}$$

Possunt autem sumi, pro quovis assignato numero, totidem in continua proportione subtripla numeri à ternario, iuxtà quorum aggregatum sumptæ fractiones dispositionis propositæ ipsum assignatam numerum superabut. Ergo propositæ fractiones in infinitum dispositæ, & aggregatæ infinitam extensionem valent implere.

Moreover, for any assigned number, one can take the same number of terms in a continuous subtriplicate proportion starting from three, based on whose sum the taken fractions of the proposed arrangement will exceed the assigned number itself. Hence, the proposed fractions, arranged into infinity and summed, are capable of filling an infinite extension.

Sit exempli gratia numerus assignatus 4: & sumantur à ternario quatuor continuè proportionales in subtripla 3, 9, 27, 81, quorum summa 120: igitur sumptæ fractiones in multitudine numeri 120 superant assignatum numerum 4; nam tres primæ superant triplum 1/3, videlicet unitatem: novem deinceps superant triplum aggregati $\frac{1}{6}$, $\frac{1}{9}$, $\frac{1}{12}$, videlicet aggregatum $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$; sed

For example, let the assigned number be 4: and from three, let four continuous proportions in subtriplicate—3, 9, 27, 81—be taken, whose sum is 120. Therefore, the fractions taken with the quantity of number 120 exceed the assigned number 4; for the first three exceed three times 1/3, that is, unity. The next nine exceed three times the aggregate of $\frac{1}{6}$, $\frac{1}{9}$, $\frac{1}{12}$, that is, the aghuiusmodi aggregatum superat unitatem, ut ostendi; ergo novem deinceps superant unitatem: & propter eamdem demonstrationem 27, & 81 subsequentes singulas unitates excedunt.

gregate $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$; but this kind of aggregate exceeds unity, as has been shown; thus, the next nine exceed unity: and for the same reason of the same demonstration, 27 and 81, following, each exceed unity.

Hinc duo Corollaria processere. Primum; quod eadem dispositio à quocunque ordinetur principio in infinitum extenditur; utpote si dispositarum fractionum prima sit $\frac{1}{5}$, & alia deinceps adhuc ipsam dispositionem propositum quemuis numerum superare posse: finitum enim est aggregatum ex iis, quæ sunt omissæ $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, & finiti ab infinito subtractiofinitum relinquere non potest.

From this follow two corollaries. First, the same arrangement, regardless of its starting point, extends infinitely; for instance, if the first of the arranged fractions is $\frac{1}{5}$, and the others thereafter still maintain the arrangement, any proposed number can be surpassed: for the aggregate of those omitted fractions (e.g., $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{5}$) is finite, and subtracting a finite quantity from infinity cannot yield a finite result.

$$\frac{1}{5}\ \frac{1}{6}\ \frac{1}{7}\ \frac{1}{8}\ \frac{1}{9}\ \frac{1}{10}\ \frac{1}{11}\ \frac{1}{12}\ \frac{1}{13}\ \&\text{ c.} \tag{11}$$

Alterum, quod infinitarum fractionum dispositio, in qua singulæ unitates à singulis numeris Arithmetice proportionalibus denominantur, pariter in infinitum extenditur. Fiat huiusmodi dispositio A, cuius primam fractionem denominet numerus B, & excessus Arithmeticæ proportionalium sit C, & sub singulis fractionibus dispositionis A, ab eodem principio fiat dispositio D, fractionum, in quibus unitates denominantium omnibus numeris à B.

Second, the arrangement of infinite fractions, in which each unit is denominated by numbers proportionally arranged arithmetically, also extends infinitely. Let there be such an arrangement A, whose first fraction is denominated by the number B, and let the excess of its arithmetic proportions be C. Under each fraction of arrangement A, let there be arrangement D, starting from the same point, of fractions in which the units of the denominators are all numbers starting from B.

$$\begin{array}{llllllll} \text{A} & \frac{1}{2} & \frac{1}{5} & \frac{1}{8} & \frac{1}{11} & \frac{1}{14} & \text{B} & 2 \\ \text{D} & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \text{C} & 3 \end{array} \tag{12}$$

Quia primi denominatores in dispositionibus A, D, sunt æquales, alter minor est quam ut ad alterum eandem habeat proportionem, quam C, ad unitatem; & colligendo secundus in dispositione A, minor est quam ut ad secundum in dispositione D, eandem habeat proportionem; sunt autem fractiones eumdem habentes numeratorem in reciproca proportione denominatorum; ergo prima, secunda, & singulæ deinceps fractiones dispositionis D, sunt minores quam ut ad primem, secundam, & singulas deinceps dispositionis A, eandem habeant proportionem, quam C, ad unitatem; & colligendo, tota dispositio D,

Since the first denominators in the arrangements A and D are equal, one is smaller than what is required for the other to have the same proportion as C to unity; and by comparing, the second in arrangement A is smaller than what is required for the second in arrangement D to have the same proportion. The fractions, however, have the same numerators in reciprocal proportion to their denominators; therefore, the first, second, and each successive fraction in arrangement D are smaller than the corresponding first, second, and successive fractions in arrangement A, to have the same proportion as C to unity. By minor est quam ut ad totam dispositionem A, eandem habeat proportionem, quam C, ad unitatem.

summing, the entire arrangement D is smaller than the entire arrangement A, having the same proportion as C to unity.

Igitur si extensionis A, quantitas assignatur; etiam eiusdem extensionis multiplam secundum numerum C, quantitatem necesse est assignari, qua infinita extensione D, sit maior; quod est absurdum, Ergo extensio infinitarum fractionum dispositionis A, est infinita.

Thus, if the quantity of extension A is assigned, it is also necessary to assign a multiple of the same extension according to the number C, with which the infinite extension D is greater, which is absurd. Therefore, the extension of the infinite fractions of arrangement A is infinite.

Dimissis igitur hisce dispositionibus quantitatis iurisdictionem superantibus, eandem contemplationem instituere cæpi de fractionibus, in quibus unitates à numeris triangularibus denominantur; an videlicet ipsæ etiam quadraturam excluderent, an potius paterentu:

Dismissing these arrangements, which exceed the satisfaction of quantity, I began to apply the same contemplation to fractions in which the units are denominated by triangular numbers: namely, whether these fractions would also exclude quadrature or rather permit it.

Factis ergo de more calculis, & instructa demonstratione, inveni dispositionis huiusmodi quadraturam esse unitatem:

Thus, after performing the usual calculations and preparing a demonstration, I found that the quadrature of such an arrangement is unity:

Units denominated by triangular numbers, which aggregated from the first are,

$$\begin{array}{ccccccc} \frac{1}{3} & \frac{1}{6} & \frac{1}{10} & \frac{1}{15} & \frac{1}{21} & \frac{1}{28} & \frac{1}{36} \\ \frac{1}{3} & \frac{2}{4} & \frac{3}{5} & \frac{4}{6} & \frac{5}{7} & \frac{6}{8} & \frac{7}{9} \end{array} \tag{13}$$

quòd aggregatæ quotlibet à prima sunt æquales numero multitudinis ipsarum denominato per numerum binario maiorem, & propterea semper unitate sunt minores eo defectu, qui iuxtà multitudinis additarum fractionum incrementum infra quolibet assignatam magnitudine diminuitur, & in infinitum evanescit.

The aggregates of any set taken from the first are equal to the number of their denominators, increased by one, divided by two, and therefore are always less than unity by the defect that diminishes as the sum of added fractions increases and vanishes into infinity.

Præterea in eadem dispositione binæ sumptæ post unitatem singularum ab unitate sunt dimidiæ:

Moreover, in the same arrangement, any two taken after unity are halves of the individual terms from unity:

$$\begin{array}{ccccccccccccc} \frac{1}{3} & & \frac{1}{6} & & \frac{1}{10} & & \frac{1}{15} & \frac{1}{22} & & \frac{1}{28} & \frac{1}{36} & & \frac{1}{45} \\ & \frac{1}{2} & & & & \frac{1}{6} & & & \frac{1}{12} & & & \frac{1}{20} & \end{array} \tag{14}$$

ergo dividendo, omnes post unitatem, unitati sunt æquales.

therefore, by division, all those after unity are equal to unity.

Tandem si eiusdem dispositionis fractiones totidem sumantur deinceps secundum numeros proportionis continuæ subduplæ à binario, videlicet 2, 4, 8, & c. Aggregatæ sunt in continuæ duplæ proportione; atqui magnitudines duplæ pro-

Finally, fractions of the same arrangement are taken in the same quantity afterward, according to the numbers of the continuously subdouble proportion from two, namely 2, 4, 8, & c. These are aggregated in a continuous douportionis aggregatæ infinitæ sunt æquales duplo primæ, cum in nostro casu prima sit dimidium unitatis, ergo propositæ fractiones aggregatæ infinitæ sunt æquales unitati.

ble proportion; and the magnitudes of the aggregated infinite double proportions are equal to twice the first, since in our case the first is half of unity. Therefore, the proposed infinite fractions, when aggregated, are equal to unity.

$$\begin{array}{ccccccccccccccccc} \frac{1}{3} & & \frac{1}{6} & \frac{1}{10} & \frac{1}{15} & & \frac{1}{21} & \frac{1}{28} & \frac{1}{36} & \frac{1}{45} & \frac{1}{55} & \frac{1}{66} & & \frac{1}{78} & \frac{1}{91} & \frac{1}{105} & \frac{1}{120} \\ & \frac{1}{2} & & & & \frac{1}{4} & & & & & & & \frac{1}{8} & & & & \end{array} \tag{15}$$

Huiusmodi sunt, quæ in primo præsentis opusculi libro demonstravi de fractionibus, in quibus unitates denominantur planis omnium numerorum ab unitate: quia enim singuli trianguli numeri singulorum huiusmodi planorum sunt dimidii, propter reciprocam proportionem.

These are the kinds of concepts I demonstrated in the first book of the present work, regarding fractions in which units are denominated by planes of all numbers starting from unity. For indeed, the individual triangular numbers of these planes are halves, due to the reciprocal proportionem.

$$\begin{array}{ccccccccccccccccc} 1 & & 2 & & 3 & & 4 & & 5 & & 6 & & 7 & & 8 & & 9 \\ & \frac{1}{2} & & \frac{1}{6} & & \frac{1}{12} & & \frac{1}{20} & & \frac{1}{30} & & \frac{1}{42} & & \frac{1}{56} & & \frac{1}{72} & \\ & 1 & & \frac{1}{3} & & \frac{1}{6} & & \frac{1}{10} & & \frac{1}{15} & & \frac{1}{21} & & \frac{1}{28} & & \frac{1}{36} & \end{array}$$

Singulæ fractiones, in quibus unitates denominantur triangulis duplæ sunt singularum, in quibus denominantur planis; & ideò utrique dispositioni eadem conveniunt demonstrationes.

Each fraction, in which the units are denominated by triangular numbers, is double that of each corresponding fraction denominated by planes; and therefore, the same demonstrations apply to both arrangements.

Ab huius fractionum dispositionis contemplatione feliciter expeditus, ad aliam progrediebar dispositionem, in qua singulæ unitates numeris quadratis denominantur. Hæc speculatio fructus quidem laboris rependit, nondum tamen effecta est solvendo, sed ingenii ditioris postulat adminiculum, ut præcisam dispositionis, quam mihimetipst proposui, summam valeat reportare.

Having successfully concluded my contemplation of this arrangement of fractions, I proceeded to another arrangement, in which each unit is denominated by square numbers. This speculation indeed rewards labor with fruit, but it has not yet been completed in solving and demands the assistance of greater ingenuity so that the precise sum of the arrangement I proposed to myself may be recovered.

We conclude our translation of the preface here.

6 Conclusion

In this document, we have discussed the work of Petri Mengoli in his treatise "Novae quadraturae arithmeticae, seu De additione fractionum" (New Arithmetic Quadratures, or on the Addition of Fractions). Mengoli's work on the harmonic series is the first known published proof that the harmonic series diverges. His work on the addition of fractions led to the statement of the Basel problem, which was solved by Leonhard Euler nearly a century later.

Mengoli's work is significant as it uses a similar approach to the methodologies of Archimedes, relying on the properties of the fractions themselves rather than the concept of limits. The translation of the preface of Mengoli's work provides insight into his methodology and his reasoning behind his work on the addition of fractions.

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